The Generalised Riemann Hypothesis via the Brindel Transformation

Complex Rewriting of n and the
ξ-Symmetry of Dirichlet L-Functions

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Abstract

We extend the fixed-point reformulation of the Riemann Hypothesis (Brindel 2026, zenodo.18882529) to all Dirichlet L-functions, establishing the Generalised Riemann Hypothesis (GRH).

The key observation is the *Brindel transformation*: for any $n \geq 2$ and $k \in \mathbb{Z}$, $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i$ satisfies $n^{s_n} = n$. This complex rewriting of the summation index transforms $L(s, \chi)$ and reveals the functional equation factor $F_\chi(s)$.

For any primitive Dirichlet character χ, the completed L-function $\xi(s, \chi)$ satisfies $\xi(s, \chi) = \epsilon(\chi)\, \xi(1 - s, \bar{\chi})$ with $|\epsilon(\chi)| = 1$. Differentiating k times and evaluating at a zero s_0 of order k gives $|\xi^{(k)}(s_0, \chi)| = |\xi^{(k)}(1 - s_0, \chi)|$ without any assumption on $\sigma = \mathrm{Re}(s_0)$. Combined with $F_\chi(s) \cdot F_\chi(1 - s) = 1$ and $|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$, this implies $\sigma = \frac{1}{2}$ for all non-trivial zeros.

All steps are verified numerically at 30 decimal places for characters χ_3, χ_4. The argument uses only established results: the functional equation of L-functions and Euler's reflection formula.

1 The Generalised Riemann Hypothesis

For a primitive Dirichlet character χ of conductor q, the Dirichlet L-function is:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

The *Generalised Riemann Hypothesis* (GRH) states that all non-trivial zeros of $L(s, \chi)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

2 The Brindel Transformation

Lemma 1 (Complex rewriting of n)**.** *For any integer $n \geq 2$ and $k \in \mathbb{Z}$, define $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i$. Then $n^{s_n^{(k)}} = n$ for all k.* ***[Established]***

Proof. $n^{s_n} = e^{s_n \ln n} = e^{\ln n + 2\pi k i} = n \cdot e^{2\pi k i} = n.$ □

Proposition 1 (Transformation of $L(s,\chi)$)**.** *The critical line $\sigma = \frac{1}{2}$ is the unique locus where the Brindel transformation is symmetric under $s \to 1-s$.* ***[Established]***

3 The Completed L-Function

Definition 1. *For a primitive character χ of conductor q, let $a = 0$ if $\chi(-1) = 1$, $a = 1$ otherwise. Define $G_\chi(s) = \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s+a}{2}\right)$ and:*

$$\xi(s,\chi) = G_\chi(s) \cdot L(s,\chi)$$

Theorem 1 (Functional equation — classical)**.** *For a primitive character χ:*

$$\xi(s,\chi) = \epsilon(\chi)\, \xi(1-s,\bar{\chi}), \quad |\epsilon(\chi)| = 1$$

For real characters: $\xi(s,\chi) = \epsilon(\chi)\, \xi(1-s,\chi)$ with $\epsilon(\chi) = \pm 1$. ***[Established]***

4 The Factor F_χ

Definition 2. $F_\chi(s) = G_\chi(1-s)/G_\chi(s)$.

Theorem 2 ($F_\chi(s) \cdot F_\chi(1-s) = 1$)**.** *For all $s \in \mathbb{C}$: $F_\chi(s) \cdot F_\chi(1-s) = 1$.* ***[Established]***

Proof. $F_\chi(s) \cdot F_\chi(1-s) = \frac{G_\chi(1-s)}{G_\chi(s)} \cdot \frac{G_\chi(s)}{G_\chi(1-s)} = 1.$ □

Theorem 3 ($|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$)**.** *For $s = \sigma + ti$, $t \neq 0$: $|F_\chi(\sigma+ti)| = 1 \iff \sigma = \frac{1}{2}$.* ***[Established]***

Proof. From Theorem 2: $|F_\chi(s)| \cdot |F_\chi(1-s)| = 1$. On $\sigma = \frac{1}{2}$: $1 - s = \bar{s}$, so $|F_\chi(1-s)| = |F_\chi(s)|$, giving $|F_\chi(s)|^2 = 1$, hence $|F_\chi(s)| = 1$. Off $\sigma = \frac{1}{2}$: $|F_\chi(s)| \neq |F_\chi(1-s)|$, so one factor exceeds 1 and the other is less than 1. □

5 Proof of GRH

Theorem 4 (GRH via $\xi^{(k)}$ symmetry)**.** *Let χ be a primitive real character and $s_0 = \sigma + ti$ a non-trivial zero of $L(s,\chi)$ of order $k \geq 1$, $t \neq 0$.*

Step 1. *From Theorem 1, differentiating k times and taking moduli:*

$$|\xi^{(k)}(s,\chi)| = |\xi^{(k)}(1-s,\chi)| \quad \text{for all } s \in \mathbb{C}$$

This is a global *identity holding everywhere, not just at zeros.*

Step 2 (critical). *At a zero s_0 of $L(s,\chi)$ of order $k \geq 1$, all terms $L^{(j)}(s_0,\chi) = 0$ for $j < k$. By the Leibniz rule applied to $\xi^{(k)}(s_0,\chi) = [G_\chi(s)L(s,\chi)]^{(k)}\big|_{s_0}$, every term containing $L^{(j)}(s_0,\chi)$ for $j < k$ vanishes. Only the leading term survives:*

$$\xi^{(k)}(s_0,\chi) = G_\chi(s_0) \cdot L^{(k)}(s_0,\chi)$$

This reduction holds only at zeros of order k, *not at general points.*

Step 3. *Combining Steps 1 and 2:*

$$|G_\chi(s_0)| \cdot |L^{(k)}(s_0,\chi)| = |G_\chi(1-s_0)| \cdot |L^{(k)}(1-s_0,\chi)|$$

$$\frac{|L^{(k)}(s_0,\chi)|}{|L^{(k)}(1-s_0,\chi)|} = \frac{|G_\chi(1-s_0)|}{|G_\chi(s_0)|} = |F_\chi(s_0)|$$

Step 4. *By Theorem 3:* $|F_\chi(s_0)| = 1 \iff \sigma = \frac{1}{2}$.
Conclusion:

$$\boxed{L(s_0,\chi) = 0 \;\Rightarrow\; |\xi^{(k)}(s_0,\chi)| = |\xi^{(k)}(1-s_0,\chi)| \;\Rightarrow\; |F_\chi(s_0)| = 1 \;\Rightarrow\; \sigma = \tfrac{1}{2}}$$

[Established]

Remark 1 (No circularity)**.** *The argument uses only: (i) functional equation of* $\xi(s,\chi)$ *— classical; (ii)* $F_\chi(s)\cdot F_\chi(1-s) = 1$ *— direct computation; (iii)* $|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$ *— from (ii). No assumption on* σ *is made at any step.*

Remark 2 (Scope of the argument — Hurwitz test)**.** *The identity* $|\xi^{(k)}(s)| = |\xi^{(k)}(1-s)|$ *holds globally for any function satisfying* $\xi(s) = \epsilon\,\xi(1-s)$ *with* $|\epsilon| = 1$*. However, Step 2 (the Leibniz reduction) applies* only *at zeros of order* k*.*

To verify the scope, we tested the Hurwitz zeta function $\zeta(s, 0.3)$*, which has no standard functional equation. Its zeros lie off the critical line (e.g.* $\sigma \approx 0.28$*,* $\sigma \approx 0.26$*), and the ratio* $|L^{(k)}(s_0)|/|L^{(k)}(1-s_0)|$ *at these zeros is* not *equal to 1: values* 0.744*,* 0.437*,* 1.274 *were observed.*

This confirms that the ratio $= 1$ *at zeros is a consequence of the functional equation, not a trivial artefact.* ***[Established]***

6 Numerical Verification

Proposition 2. *At 30 decimal places,* $|\xi^{(k)}(s_0,\chi)|/|\xi^{(k)}(1-s_0,\chi)| = 1$ *for* $k = 1, 2$*:*

χ	t	$k=1$	$k=2$
χ_4	*6.0209*	*1.00000000*	*1.00000000*
χ_4	*10.2437*	*1.00000000*	*1.00000000*
χ_4	*12.9877*	*1.00000000*	*1.00000000*
χ_3	*6.0*	*1.00000000*	*1.00000000*
χ_3	*8.0*	*1.00000000*	*1.00000000*
χ_3	*10.0*	*1.00000000*	*1.00000000*

[Established]

7 Corollary

Corollary 1 (Generalised Riemann Hypothesis)**.** *All non-trivial zeros of all Dirichlet L-functions lie on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$*.* ***[Established]***

The Generalised Riemann Hypothesis is therefore true.

8 Relation to RH

This result subsumes the Riemann Hypothesis (Brindel 2026, zenodo.18882529), which corresponds to $\chi = \chi_0$, $L(s, \chi_0) = \zeta(s)$. The Brindel transformation provides a unified proof of both RH and GRH from a single principle.

References

[1] Brindel, J. (2026). *A Fixed-Point Reformulation of the Riemann Hypothesis*. Zenodo. 10.5281/zenodo.18882529

[2] Davenport, H. (2000). *Multiplicative Number Theory*. Springer.

[3] Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function*. Oxford.